X. *On the Thermo-dynamic Theory of Steam-engines with dry saturated Steam, and its application to practice.* By WILLIAM JOHN MACQUORN RANKINE, *C.E., LL.D., F.R.SS.L. & E., Pres. Inst. Eng. Scot., Regius Professor of Civil Engineering and Mechanics in the University and College of Glasgow.*

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Introduction.

IT was demonstrated independently from the laws of Thermo-dynamics, by Professor CLAUSIUS and the author of this paper, in 1849*, that when steam or other saturated vapour in expanding performs work by driving a piston, and receives no heat from without during that expansion, a portion of it must be liquefied.

That theoretical conclusion has since been amply confirmed by experience in actual steam-engines; for it has been ascertained that the greater part of the liquid water which collects in unjacketed cylinders, and which was once supposed to be wholly carried over in the liquid state from the boiler (a phenomenon called "priming"), is produced by liquefaction of part of the steam during its expansion; and also that the principal effect of the "*jacket*," or annular casing enveloping the cylinder, filled with hot steam from the boiler, which was one of the inventions of WATT, is to prevent that liquefaction of the steam in the cylinder.

That liquefaction does not, when it first takes place, directly constitute a waste of heat or of energy; for it is accompanied by a corresponding performance of work. It does, however, afterwards by an indirect process, diminish the efficiency of the engine; for the water which becomes liquid in the cylinder, probably in the form of mist and spray, acts as a distributer of heat and equalizer of temperature, abstracting heat from the hot and dense steam during its admission into the cylinder, communicating that heat to the cool and rarefied steam which is on the point of being discharged, and thus lowering the initial pressure and increasing the final pressure of the steam; but lowering the initial pressure much more than the final pressure is increased. Accordingly, in all cases in which steam is expanded from a high down to a low pressure, it has in practice been found necessary to envelope the cylinder in a steam-jacket†. The liquefaction which would otherwise have taken place in the cylinder, takes place in the jacket instead, where the presence of the liquid water produces no bad effects, and that water is returned to the boiler.

In double-cylinder engines, where the expansion of the steam begins in a smaller

* POGGENDORFF's 'Annalen,' 1850; Edinburgh Transactions, vol. xx.

† Unless the steam is superheated (Sept. 1859).

cylinder and finishes in a larger, the usual practice is to have steam-jackets round both cylinders; but in a few examples in which the smaller cylinder alone is jacketed, the liquefaction is found to be almost wholly prevented, showing that the steam during its passage from the smaller to the larger cylinder receives sufficient heat, either directly from the small cylinder or indirectly by conduction from the smaller to the larger cylinder (which is in close contact with the small cylinder), to prevent any appreciable portion of it from condensing.

The only *exact* formulæ hitherto published for the work performed by the steam on the piston, viz. those contained in a paper on thermo-dynamics by the author of this paper, which was received by the Royal Society in 1853 and published in the Philosophical Transactions for 1854, and those contained in a paper by Professor CLAUSIUS, published in POGGENDORFF'S 'Annalen' for 1856 (the same results having been independently arrived at in both papers), are adapted to cylinders *without steam-jackets*.

It is obviously desirable that exact formulæ should be deduced from the principles of thermo-dynamics, for the action of steam in jacketed cylinders also; and the present paper is intended to supply that want. In the first place, those fundamental equations of thermo-dynamics, to which it is afterwards necessary to refer, are briefly recapitulated: then the exact formulæ for the action of steam, and the corresponding expenditure of heat, in jacketed cylinders are deduced from them, and exemplified by numerical results:—then is explained a convenient approximation to the exact formulæ, founded on the facts, that within the limits usual in practice, the pressure of saturated steam varies nearly as the seventeenth power of the sixteenth root of its density; and that the expenditure of heat in a jacketed cylinder is nearly equal to fifteen and a half times the product of the initial pressure and volume of the steam:—and, lastly, are given examples of the application of the formulæ to the engines of three steam-vessels recently experimented on by the author, and a comparison of the results of the formulæ with those of experiment.

Summary of previously-known Principles and Formulæ.

The following is the GENERAL EQUATION OF THERMO-DYNAMICS which has been known since 1849–50*:—

$$H=\int t\,d\phi. \qquad (1.)$$

H denotes the *quantity of heat* which must be communicated to a mass of matter in order to make it undergo a given series of changes of volume and elastic pressure, expressed in *foot-pounds of energy*, of which 772 are equivalent to one degree of FAHRENHEIT in one pound of water, as proved by Mr. JOULE†.

t is *absolute temperature*, reckoned from the absolute zero which corresponds to total privation of heat. It has from the first been conjectured that the scale of absolute temperature coincides with that of a perfect-gas thermometer, the absolute zero being

* Edinburgh Transactions, vol. xx.; and POGGENDORFF'S 'Annalen,' 1850.

† Philosophical Transactions, 1850.

the temperature corresponding to total absence of elastic pressure. On that supposition the specific heat of air under constant pressure was predicted in 1850 as being probably 0·24 of that of water, or thereabouts*; and M. REGNAULT, in 1853, ascertained it by experiment to be 0·238. The long series of experiments by Messrs. JOULE and THOMSON on the thermic effects of currents of elastic fluids†, have proved still more conclusively that if the scale of absolute temperature and that of the perfect-gas thermometer differ, it must be by quantities so small that they have not yet been measured.

The expansion of a perfect gas from 32° FAHR. to 212° FAHR. being in the ratio 1:1·365, the absolute zero is

$$\frac{180^\circ}{0{\cdot}365}=493^\circ{\cdot}2 \text{ FAHR.}$$

below the temperature of melting ice; or

$$t \text{ in degrees of FAHR.}=461^\circ{\cdot}2+\mathrm{T},$$

T being the temperature on the ordinary FAHRENHEIT'S scale.

ϕ is a function which remains constant when the mass under consideration either performs work by expansion, or undergoes compression, without receiving or emitting heat. Its value is

$$k.\text{hyp. log } t+\int\frac{dp}{dt}\,dv; \qquad \text{(2.)}$$

where k is the real specific heat of the substance, expressed in foot-pounds of energy per degree of temperature; p is the elastic pressure of the mass per unit of area when it occupies the volume v; the differentiation $\frac{dp}{dt}$ is performed on the supposition of v being constant, and the integration on the supposition of t being constant.

Another form of the function ϕ, which is convenient in certain calculations, is as follows:—

$$\phi=\left(k+\frac{p_0v_0}{t_0}\right)\text{hyp. log } t-\int_0^p\frac{dv}{dt}\,dp, \qquad \text{(3.)}$$

where $\frac{p_0v_0}{t_0}$ is the constant value of the ratio $\frac{pv}{t}$ for the substance under consideration, in the perfectly gaseous state‡.

The function ϕ is sometimes called the *Thermo-dynamic function.*

One of its properties is as follows:—that when the series of changes of pressure and volume to which the integration of equation 1 is applied constitute a *cycle*, so that the mass returns in the end to its primitive volume and pressure, then for a complete cycle

$$\int(t_1-t_2)d\phi=\int(\phi_1-\phi_2)dt=\int(p_1-p_2)dv=\int(v_1-v_2)dp, \qquad \text{(4.)}$$

* Edinburgh Transactions, vol. xx.

† Philosophical Transactions, *passim.*

‡ $k+\frac{p_0v_0}{t_0}$ is the specific heat of the substance *under constant pressure,* in the perfectly gaseous state, expressed in units of energy.

in which t_1 and t_2 are the two values of t corresponding to one given value of ϕ, and the other similar symbols have analogous meanings.

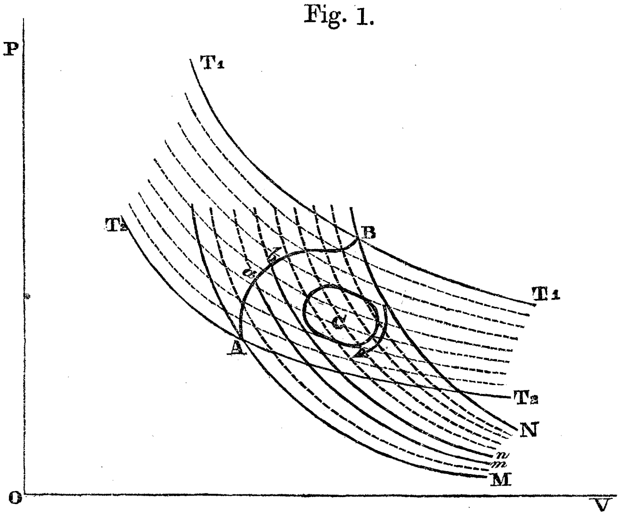
Equation 4, expressed in words, means—*The heat which disappears during a cycle of operations is equal to the work performed.* Another mode of expressing the principle of equation 4 is as follows:—

$$dt.d\phi=dp.dv. \quad \ldots\ldots\ldots \quad (4\text{ A.})$$

The following is a summary of that graphic representation of the General Equation of Thermo-dynamics which was first demonstrated in the Philosophical Transactions for 1854.

In fig. 1, let OP, OV be rectangular axes of coordinates; and let ordinates measured parallel to OP and OV respectively represent pressures and volumes, so that areas

Fig. 1.



represent quantities of energy. Let the coordinates of a line of any figure, such as AB, represent a series of changes of pressure and volume undergone by an elastic body. Let AM, BN be curves traversing A and B, of the class called *adiabatic* curves, or curves of no transmission; that is, curves whose coordinates show the law of variation of the pressure and volume of the body when it neither receives nor emits heat. Those curves are indefinitely extended both ways, and OV is their asymptote.

Then *the heat which the body receives during the change from* A *to* B, *is represented by the area contained between the line* AB *and the curves* AM *and* BN, *indefinitely extended in the direction* OV.

To show the connexion between this and the algebraical expression of the same law in equation 1, let ϕ be a function which is constant for a given adiabatic curve, and of such a nature that $dt.d\phi=dp.dv$. Divide the area MABN into an indefinite number of indefinitely-narrow bands by a series of adiabatic curves. Let *mabn* be one of these bands. Let the difference of the values of the function ϕ for the curves *am*, *bn*, be $d\phi$; let t be the absolute temperature corresponding to the element *ab* of the line AB; then—

$$\text{area } mabn=td\phi; \quad \text{area MABN}=\int td\phi.$$

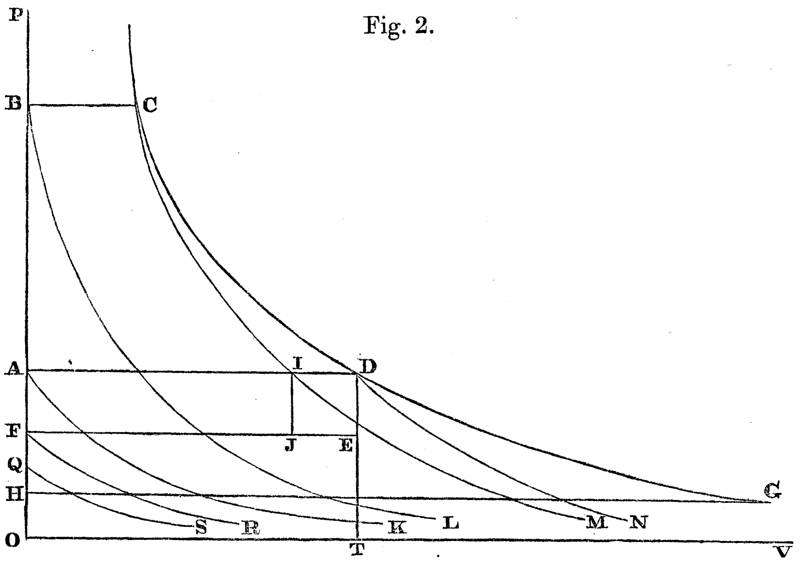
Let $T_1 T_1$, $T_2 T_2$, and the dotted lines between them, be *isothermal lines*, each of which by its coordinates represents the relation between the pressure and volume of the body for a particular uniform temperature. Then the scale of Absolute Temperatures is such, that a series of isothermal lines, corresponding to a series of equal divisions upon that scale, divides the band between any pair of adiabatic curves into equal areas.

A *cycle of changes* is represented by a closed figure, such as C; and the area of that figure represents the heat transformed into mechanical energy, or the mechanical energy transformed into heat, according as the cycle of changes takes place in the direction represented by the arrow, or in the contrary direction. That area is the quantity expressed by equation 4.

The following are the results of applying the general equation of thermo-dynamics to fluids which are in the act of changing from the liquid to the gaseous state, or nascent vapours.

In fig. 2, let the line $\overline{BC}$ parallel to OV represent the increase of volume which a

Fig. 2.



given fluid mass undergoes in changing from the liquid to the gaseous state under a pressure represented by the ordinate $\overline{OB}=p$. Through B and C draw a pair of indefinitely extended adiabatic curves, BL, CIM; then the area LBCM represents the *latent heat of evaporation* of the fluid mass under the pressure p.

To find the algebraical expression for that latent heat, it is to be considered, that in applying to this case the formula $H=td\phi$, t is the absolute temperature of the *boiling-point* corresponding to the pressure p, and is *constant*; so that

$$H=t(\phi_c-\phi_b);$$

ϕ_c and ϕ_b being the values of the thermo-dynamic functions for the curves CM and BL respectively. It is next to be considered, that because t is constant, the term of equation 2 which depends on t alone, is the same in ϕ_c and ϕ_b, so that it disappears from their difference, which consequently becomes simply

$$\phi_c - \phi_b = \int \frac{dp}{dt} \cdot dv;$$

and that because $\frac{dp}{dt}$ is constant during the evaporation, the value of the above integral is $v \cdot \frac{dp}{dt}$, where v denotes the *excess of the volume of the vapour above that of the liquid*, or *the increase of volume in the act of evaporation*; so that, finally, the latent heat of evaporation under the pressure p, in units of mechanical energy, is

$$H = vt \cdot \frac{dp}{dt}. \quad \text{(5.)}$$

Now the relation between the pressure p and boiling-point t for water and various other fluids has been ascertained by the experiments of M. REGNAULT, and expressed in tables and formulæ; and for water, the relation between the boiling-point t and latent heat of evaporation H, has also been ascertained by M. REGNAULT'S experiments in ordinary units of heat, which by the aid of "JOULE'S equivalent" can be reduced to units of energy; consequently the INCREASE OF VOLUME OF ONE POUND OF WATER IN EVAPORATING is given by the equation

$$v = \frac{H}{t \cdot \frac{dp}{dt}}. \quad \text{(6.)}$$

The volume of a given weight of liquid water at all ordinary pressures is so small, as compared with that of the same weight of steam, that for practical purposes v may without material error be considered as representing the *volume of one pound of steam.*

The latent heat of evaporation of one pound of water at the boiling-point t in foot-pounds of energy, is given with sufficient accuracy by the formulæ

$$H = a - bt, \quad \text{(7.)}$$

where $a = 1109550$ foot-pounds,
and $b = 540{\cdot}4$ foot-pounds per degree of FAHRENHEIT.

The pressure of saturated steam is given with great exactness, at all boiling-points at which it has yet been measured, by the formula

$$\text{com. log}\, p = A - \frac{B}{t} - \frac{C}{t^2} \quad \text{(8.)}$$

(first published in the Edinburgh Philosophical Journal, 1849). The values of the constants A, B, C, given as revised, in the Philosophical Magazine for December 1854, are as follows:—

For pounds on the square foot, $A = 8{\cdot}2591$.
For pounds on the square inch, $A = 6{\cdot}1007$.
For absolute temperatures in degrees of FAHRENHEIT,

$$\log B = 3{\cdot}43642; \ \log C = 5{\cdot}59873.$$

In using equation 6, the unit of volume and the unit of pressure to be employed depend on each other. The following are examples:—

Unit of Pressure.	*Unit of Volume.*
Pound on the square foot.	Cubic foot.
Pound on the square inch.	Prism one foot long by one inch square, or $\frac{1}{144}$ cubic foot.

In either of the above cases, quantities of energy and of heat are expressed in foot-pounds.

The latent heat of so much steam as occupies a unit of volume more in the gaseous state than it did in the liquid state is obviously

$$t\frac{dp}{dt}=p\left\{\frac{\mathrm{B}}{t}+\frac{2\mathrm{C}}{t^2}\right\}\text{ hyp. log }10 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.)$$

(hyp. log 10=2·3026 nearly).

In every case of the working of steam which occurs in practice, the volume of the liquid water is so small a fraction of the volume of the steam, that it may be neglected without sensible error. When this is done, the indicator-diagram of a steam-engine working perfectly, and without transmission of heat to or from the steam in the cylinder, may be represented in the following manner.

In fig. 2, let $\overline{\mathrm{OB}}=p_1$ represent the "pressure of admission" at which the steam is admitted into the cylinder; t_1 the corresponding boiling-point:—

$\overline{\mathrm{BC}}=v_1$ the volume of one pound of steam when admitted:—

$\overline{\mathrm{OA}}=p_2$ the final pressure of the steam in the cylinder at the end of the expansion; t_2 the corresponding boiling-point:—

$\overline{\mathrm{OF}}=p_3$ the "pressure of exhaustion" at which the steam is expelled from the cylinder; t_3 the corresponding boiling-point.

Let AID and FJE be parallel to OV and BC. Draw the adiabatic curves CIM, BL, FR. Then CI will be the *curve of expansion* of the steam, and $\overline{\mathrm{AI}}=s_2$ will represent the volume occupied by one pound at the end of the expansion. The work of one pound of steam on the piston will be represented by the area FABCIJF, consisting of the parts

$$\mathrm{ABCIA}=\int_{p_2}^{p_1} s\,dp\text{; and AIJF}=s_2(p_2-p_3). \quad . \quad . \quad . \quad . \quad . \quad (10.)$$

The symbol s is used to denote the volume occupied by one pound of the mixture of steam and liquid water which the cylinder contains at any given time during the expansion, s_2 being the final value of that volume.

Let CDG be a curve whose ordinates parallel to OV represent the volumes of one pound of dry saturated steam at the pressures represented by its ordinates parallel to OP. Then $\mathrm{AD}=v_2$ is the volume which one pound of steam would occupy at the end of the expansion, but for the partial liquefaction; and the proportion of the steam liquefied is

$$\frac{\overline{\text{ID}}}{\overline{\text{AD}}}=\frac{v_2-s_2}{v_2}. \quad \ldots \ldots \quad (11.)$$

The expenditure of heat per pound of steam (supposing the feed-water to be supplied to the boiler at the temperature of exhaustion, t_3) is represented by the indefinitely-prolonged area RFBCM, and consists of two parts,—

The sensible heat RFBL$=\text{J}(t_1-t_3)$, where J is the specific heat of liquid-water, 772 lb. per degree of FAHRENHEIT, and the latent heat LBCM$=$H (see eq. 7.) (12.)

If the feed-water is supplied to the boiler, not at the temperature of exhaustion, t_3, but at some lower temperature, t_4, the latter temperature must be substituted for the former in the formulæ.

The application of these principles to steam-engines without jackets has been fully explained and exemplified in the paper already referred to in the Philosophical Transactions for 1854.

Calculations respecting such engines in ordinary cases are facilitated by approximate formulæ, founded on the fact that within the usual limits of pressure, viz. with $p_1=$ from one to twelve atmospheres, the coordinates of the curve CM are related by the equation

$$p \propto s^{-\frac{10}{9}} \text{ nearly*}. \quad \ldots \ldots \quad (13.)$$

The expenditure of heat can be roughly computed to within about $\frac{1}{30}$ of the truth, by the formula

$$v_1(13\tfrac{1}{3}p_1+4000)\text{*}, \quad \ldots \ldots \quad (14.)$$

v_1 being in cubic feet and p_1 in pounds on the square foot*.

Theory of the Work, Heat, and Efficiency of Dry Saturated Steam.

In the following investigation it is assumed that the steam while expanding receives just enough of heat to prevent any part of it from condensing, without super-heating it. This assumption is founded on the fact, that dry steam is a bad conductor of heat as compared with liquid water, or with cloudy steam, and that after cloudy steam has received enough of heat to make it dry or nearly dry, it will probably receive little more.

The assumption is justified by the fact, that its results are confirmed by experiment.

The symbol v is used to denote the volume of one pound of steam in *cubic feet*, and the symbol p to denote pressure in *pounds on the square foot*; so that pressure in pounds on the square inch is denoted by $\frac{p}{144}$.

In fig. 2, as before, let CDG be the curve whose coordinates represent the volumes and pressures of dry saturated steam.

* Manual of Applied Mechanics, Art. 656.

Let $\overline{OB}=p_1$, and $\overline{BC}=v_1$, represent the pressure and volume of admission, and t_1 the corresponding absolute temperature:—

Let $\overline{OA}=p_2$, and $\overline{AD}=v_2$, represent the pressure and volume at the end of the expansion, and t_2 the corresponding absolute temperature; then

$\frac{v_2}{v_1}=r$ is the *ratio of expansion*, and

$\frac{v_1}{v_2}=\frac{1}{r}$ the *effective cut-off.*

Let $\overline{AF}=p_3$ be the pressure of exhaustion, and t_3 the corresponding absolute temperature;

Let t_4 be the absolute temperature of the feed-water; and

Let $OQ=p_4$ be the corresponding pressure.

The work of one pound of steam is represented by the area of the diagram ABCDEFA, consisting of

$$\left.\begin{array}{l}\text{the area ABCDA}=\int_{p_2}^{p_1} v\,dp, \text{ and} \\ \text{the area ADEF } =v_2(p_2-p_3);\end{array}\right\} \quad \ldots \ldots \quad (15.)$$

while the expenditure of heat per pound of steam is represented by the area contained between the line QFABCD, and the two indefinitely extended adiabatic curves, QS, DN, and may be distinguished into the following parts:—

$$\left.\begin{array}{ll}\text{the sensible heat SQBL} & =J(t_1-t_4) \\ \text{the latent heat of evaporation, LBCM} & =H_1; \\ \text{the latent heat of expansion} & =\text{MCDN.}\end{array}\right\} \quad \ldots \ldots \quad (16.)$$

Thus it appears that the work of one pound of dry saturated steam exceeds that of one pound of steam which expands from the same initial pressure to the same final pressure without receiving heat, by an amount represented by the area JICDEJ, while the expenditure of heat is greater by the quantity represented by the area MCDN.

To find the area ABCDA, which represents part of the work, the value of v corresponding to any value of p is to be taken from equation 6, and that of H, the corresponding latent heat of evaporation, from equation 7, giving

$$v=\frac{a-bt}{t\frac{dp}{dt}},$$

which being multiplied by $\frac{dp}{dt}dt$, and integrated between t_1 and t_2, the initial and final temperatures of the expanding steam, we obtain for the area ABCDA,

$$\int_{p_2}^{p_1} v\,dp=\int_{t_2}^{t_1}\left(\frac{a}{t}-b\right)dt=a\,\text{hyp. log}\,\frac{t_1}{t_2}-b(t_1-t_2); \quad \ldots \ldots \quad (17.)$$

to which adding the rectangle ADEFA, the WORK OF ONE POUND OF STEAM is found to be

$$W=\int_{p_2}^{p_1} v\,dp+v_2(p_2-p_3)=a\ \text{hyp.}\log\frac{t_1}{t_2}-b(t_1-t_2)+v_2(p_2-p_3); \quad . \quad (18.)$$

in which

$a=1109550$ foot-pounds; $b=540{\cdot}4$ foot-pounds per degree of FAHRENHEIT.

The MEAN EFFECTIVE PRESSURE, or work per unit of volume traversed by the piston, is

$$w=W\div v_2. \quad \ldots\ldots \quad (18\text{A}.)$$

The heat expended per pound of steam, by a different mode of division from that given in the formulæ 16, is computed as follows:

Part of the sensible heat, $SQAK=J(t_2-t_4)$;

Latent heat of evaporation at the temperature t_2, $KADN=H_2=a-bt_2$;

Work performed between the temperatures t_1 and t_2, $ABCDA=\int_{p_2}^{p_1} v\,dp$ as in equation 17.

The addition of those quantities gives for the whole EXPENDITURE OF HEAT PER POUND OF STEAM in foot-pounds of energy,

$$\mathfrak{H}=J(t_2-t_4)+a-bt_2+\int_{p_2}^{p_1} v\,dp=J(t_2-t_4)+a\left(1+\text{hyp.}\log\frac{t_1}{t_2}\right)-bt_1 \quad \ldots \quad (19.)$$

($J=772$ foot-pounds per degree of FAHRENHEIT).

The heat expended per unit of space traversed by the piston is equivalent to a pressure whose intensity is

$$h=\mathfrak{H}\div v_2. \quad \ldots\ldots \quad (19\text{A}.)$$

The EFFICIENCY of the steam is the ratio

$$E=W\div\mathfrak{H}=\frac{w}{h}, \quad \ldots\ldots \quad (20.)$$

of the work performed by the steam on the piston to the heat expended on the steam; and that ratio having been determined, the available heat of a pound of fuel may be computed from the indicated work per pound of fuel, or *vice versâ*, by means of the equation,

$$\frac{\text{available heat}}{\text{indicated work}}=\frac{1}{E}. \quad \ldots\ldots \quad (21.)$$

In the practical use of equations 18, 18A, 19, 19A, 20, and 21, the usual data are,—the initial pressure p_1, the ratio of expansion r, the pressure of exhaustion p_3, and the temperature of the feed-water t_4. From p_1, by the aid of equations 6, 7, 8, 9, or of tables, are to be found t_1 and v_1. Then

$$rv_1=v_2;$$

and from v_2, by the aid of the same equations, or of tables, are to be found t_2 and p_2, and thus are completed the data for the use of equations 18 and 19.

Let $\overline{OH}=p_0$ represent the pressure, and $\overline{HG}=v_0$ the volume, of a pound of steam at some standard temperature, such as that of melting ice ($t_0=32°+461°{\cdot}2=493°{\cdot}2$ FAHR.); and let

$$U=\int_{p_0}^{p} v dp = a \text{ hyp. log} \frac{t}{t_0} - b(t-t_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22.)$$

be the area contained between HG and another parallel ordinate of the curve CDG, corresponding to the absolute temperature t.

Then by the aid of tables of the function U, the equations 18 and 19 can be put into the following form:—

$$\left.\begin{array}{l} W=U_1-U_2+v_2(p_2-p_3); \\ \mathfrak{h}=U_1-U_2+J(t_2-t_4)+a-bt_2 \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23.)$$

Tables of the values of p, v, and U, for every ninth degree of FAHRENHEIT'S scale from 32° to 428° above the ordinary zero, have been calculated, and are now being printed*. As an example of the results contained in them, the following extract is given for every thirty-sixth degree from 104° to 392° FAHRENHEIT.

Extract from Table.

T.	t.	p.	v.	U.
104°	565°·2	152·6	312·8	112290
140	601·2	414·3	122·0	161340
176	637·2	987·6	53·92	206410
212	673·2	2116·4	26·36	247950
248	709·2	4152	14·00	286290
284	745·2	7563	7·973	321780
320	781·2	12940	4·816	354670
356	817·2	20990	3·057	385200
392	853·2	32520	2·025	413580

For the purpose of interpolating intermediate numbers in such tables, the *logarithms* of p and v are more convenient than those numbers themselves, as their successive differences are more nearly uniform.

Approximate Formulæ.

As the formulæ of the preceding section require in their use a considerable amount of calculation, and in some cases the solution of transcendental equations by trial and error (unless special tables are at hand), it is desirable to have, for the purpose of solving ordinary practical problems, approximate formulæ of a more simple kind. Those which will now be explained were arrived at by a process of trial, based upon a table of the results of the exact formulæ; and their agreement with the exact formulæ, and with experiment, has been tested for initial pressures ranging from 30 to 120 lbs. on the square inch, and for ratios of expansion ranging from *four* to *sixteen.*

* In a work "On the Steam-Engine and other Prime Movers."

They may therefore be applied with confidence to engines working within those limits, and probably somewhat above them; but for pressures much exceeding 120 lbs. on the square inch, and ratios of expansion exceeding 16, it is advisable for the present to use the exact formulæ.

The foundation of the approximate formulæ for work, and for mean effective pressure, is the fact, that for pressures not exceeding 120 lbs. on the square inch, or 17280 lbs. on the square foot, the equation of the curve CDG is very nearly

$$p \propto v^{-\frac{17}{16}}. \qquad (24.)$$

This equation is very convenient in calculation, because the sixteenth root can be extracted, with great rapidity, to a degree of accuracy sufficient for practical purposes, by the aid of a table of squares alone; and by a little additional labour, without any tables whatsoever.

Let r, as before, be the ratio of expansion; then evidently,

Final pressure $p_2 = p_1 . r^{-\frac{17}{16}}$; . . . . (25.)

Gross work per lb. steam = area OBCDTO $= \mathrm{W} + p_3 v_2 = \left. \begin{array}{l} \int_{p_2}^{p_1} v dp + p_2 v_2, \\ = p_1 v_1 (17 - 16 r^{-\frac{1}{16}}) = p_1 v_2 (17 r^{-1} - 16 r^{-\frac{17}{16}}) \end{array} \right\}$; . . . (26.)

Effective work per lb. steam = area FBCDEF $= \mathrm{W} = \left. \begin{array}{l} \int_{p_2}^{p_1} v dp + (p_2 - p_3) v_2 \\ = v_2 \{ p_1 (17 r^{-1} - 16 r^{-\frac{17}{16}}) - p_3 \} \end{array} \right\}$; . . . (27.)

Mean gross pressure $= \frac{\mathrm{W}}{v_2} + p_3 = p_1 (17 r^{-1} - 16 r^{-\frac{17}{16}})$; . . . . (28.)

Mean effective pressure, or work per cubic foot, $w = \frac{\mathrm{W}}{v_2} = p_1 (17 r^{-1} - 16 r^{-\frac{17}{16}}) - p_3$. . (29.)

It is evident, that if the pressure of exhaustion p_3 be given, and any two out of the following three quantities,—the initial pressure p_1, the mean effective pressure w, the ratio of expansion r,—the fourth quantity can be calculated directly, if it is one or other of the pressures, p_1, w, or by approximation, if it is the ratio of expansion r.

The approximate formula for the expenditure of heat, in foot-pounds per pound of steam, which has been found by trial to agree very closely with the exact formula within the limits already specified, and when the feed-water is supplied at a temperature of from 100° to 120° FAHR., is as follows:

$$\mathfrak{H} = 15\tfrac{1}{2} . p_1 v_1 = \frac{15\frac{1}{2} p_1 v_2}{r}; \qquad (30.)$$

so that the *heat expended per cubic foot of space traversed by the piston*, or, the pressure per square foot of piston to which the expenditure of heat is equivalent, is

$$h = \frac{\mathfrak{H}}{v_2} = \frac{15\frac{1}{2} . p_1}{r}. \qquad (31.)$$

This gives for the *efficiency*,

$$\mathrm{E} = \frac{\mathrm{W}}{\mathfrak{H}} = \frac{w}{h} = \frac{17 - 16 r^{-\frac{1}{16}}}{15\frac{1}{2}} - \frac{r p_3}{15\frac{1}{2} p_1}; \qquad (32.)$$

by means of which, when the work of one pound of coal is known, its available heat can be computed, and *vice versâ*, as with the exact formula.

Tables of the ratios given by the equations from 25 to 29 for various ratios of expansion have been computed, and are in course of being printed. The following are examples of the results contained in them:—

Expansion r.	Effective cut-off $1 \div r$.	Mean gross pressure $\div$ initial pressure, $(w+p_3)\div p_1$.
20	0·05	0·186
10	0·10	0·314
$6\frac{2}{3}$	0·15	0·417
5	0·20	0·505
4	0·25	0·582
$3\frac{1}{3}$	0·30	0·648
$2\frac{6}{7}$	0·35	0·707
$2\frac{1}{2}$	0·40	0·756
$2\frac{2}{9}$	0·45	0·800
2	0·50	0·840
$1\frac{9}{11}$	0·55	0·874
$1\frac{2}{3}$	0·60	0·900
$1\frac{7}{13}$	0·65	0·929
$1\frac{3}{7}$	0·70	0·945
$1\frac{1}{3}$	0·75	0·960
$1\frac{1}{4}$	0·80	0·976

Comparison of Theory with Experiment.

In comparing the results of formulæ for the expansive working of steam with those of the indicator-diagrams of engines, it is not to be expected that the indicated pressures corresponding to particular volumes during, or at the end of, the expansion, will closely agree with those given by calculation; because considerable deviations of the line marked on the diagram, alternately upwards and downwards, arise from the friction of the indicator, from elastic vibration of the indicator-spring, and from oscillations of the steam itself. In the course of a complete stroke, however, those deviations neutralize each other, so that the indicated *mean effective pressure*, if the theory is sound, ought to agree with that given by theory within the limit of errors of observation.

About half a pound on the square inch, or 72 lbs. on the square foot, may be considered as an ordinary limit of error in indicator-diagrams.

Two examples of the application of the exact formulæ, and four of the application of the approximate formulæ, to actual engines, are annexed; and the results of the formulæ are compared with those of experiment.

EXAMPLE I.—Paddle-steamer of 820 tons displacement, with a pair of double-cylinder engines of 744 indicated horse-power.

DATA:—

	Bottom of cylinders.	Top of cylinders.
	lb. per square inch.	
$p_1 \div 144$	33·7	34·3
$p_3 \div 144$	4·0	4·0
r	$4\frac{1}{6}$	$6\frac{1}{4}$

$T_4 = t_4 - 461{\cdot}2 =$ about 104° FAHR.

RESULTS by exact formula:—

	Bottom of cylinders.	Top of cylinders.
$v_2 = rv_1$	50·375	74·4
$p_2 \div 144$	7·367	4·867
W	109552	117338
$\frac{W}{144v_2} = w \div 144$	15·1	10·95
Mean	13·03	

OBSERVED mean effective pressure, lb. on the inch . . . 13·10

Difference . . . $-$ 0·07

$\mathfrak{H}$	906989	925678
$\frac{\mathfrak{H}}{144v_2} = h \div 144$	125	86·4
Mean	105·7	
$E = \frac{w}{h}$	0·121	0·127

$$\text{Mean efficiency} = \frac{\text{mean } w}{\text{mean } h} = 0{\cdot}123.$$

EXAMPLE I., calculated by approximate formula.

DATA:—

		lb. per inch.
Mean . . .	$p_1 \div 144$	34
Mean . . .	$p_3 \div 144$	4
Mean . . .	$\frac{1}{r}$	0·2

RESULTS:—

	lb. per inch.
$w \div 144$, Calculated	13·17
Observed	13·10
Difference	+ 0·07
$h \div 144$	105·4
E	0·125

EXAMPLE II.—Screw-steamer of about 700 tons displacement (?), with engine of 226 indicated horse-power.

DATA:—

	Bottom of cylinders.	Top of cylinders.
	lb. on the square inch.	
$p_1 \div 144$	$108\frac{1}{2}$	104
$p_3 \div 144$	3·3	4·0
r	16	14

$T_4 = t_4 - 461 \cdot 2 = 122°$ FAHR. nearly.

RESULTS by exact formula:—

	Bottom of cylinders.	Top of cylinders.
$v_2 = rv_1$	64·27	58·52
$p_2 \div 144$	5·6	6·3
W	191437	182108
$\frac{W}{144 v_2} = \frac{w}{144}$	20·7	21·6
	Mean	21·15

OBSERVED mean effective pressure, lb. on the inch . . 21·0

Difference . . . + 0·15

$\mathfrak{H}$	975301	966524
$\frac{\mathfrak{H}}{144 v_2} = h \div 144$	105	115
	Mean	110
$E = w \div h$	0·196	0·188

Mean efficiency $= \frac{\text{mean } w}{\text{mean } h}$ . . . . . 0·192

EXAMPLE II., calculated by approximate formula.

DATA:—

	lb. per inch.
Mean . . . $p_1 \div 144$	$106\frac{1}{4}$
Mean . . . $p_3 \div 144$	3·65
Mean . . . $\frac{1}{r}$	0·067

RESULTS:—

	lb. per inch.
$w \div 144$, Calculated	21·05
Observed	21·00
Difference	+ 0·05
$h \div 144$	110
E	0·192

EXAMPLE III.—Paddle-steamer of 1100 tons displacement, with a pair of engines of 1176 indicated horse-power.

In this and the following example, some of the data were not obtained with sufficient precision to make it worth while to use the exact formulæ; and therefore the approximate formulæ alone are employed.

DATA:—

		lb. per inch.
Mean . . .	$p_1 \div 144$	38
Mean . . .	$p_3 \div 144$	$3\frac{1}{2}$
Mean . . .	$\frac{1}{r}$	0·15

RESULTS:—

		lb. per inch.
$w \div 144$,	Calculated	12·35
	Observed	12·2
	Difference	+ 0·15
$h \div 144$		88·3
E		0·14

EXAMPLE IV.—The same steamer as in Example III., with only one of her two boilers at work; indicated horse-power, 568.

DATA:—

		lb. per inch.
Mean . . .	$p_1 \div 144$	32
Mean . . .	$p_3 \div 144$	3
Mean . . .	$\frac{1}{r}$	0·1

RESULTS:—

		lb. per inch.
$w \div 144$,	Calculated	7·05
	Observed	7·0
	Difference	+ 0·05
$h \div 144$		49·6
E		0·142